Assignment 2: Counting and Probability

Instructions. All answers must be **handwritten** on separate paper. Label each answer with its problem and part number. For a calculation, show the formula or main calculation and the result. For a definition, include its mathematical conditions. For a proof, give the logical steps. For a true/false question, give your judgment and a brief reason; a counterexample is sufficient to refute a claim. Exact expressions are acceptable unless a decimal approximation is requested. AI tools are permitted under the course policy, but you are responsible for understanding and checking everything you submit.

1. Order and repetition

Calculate the number of outcomes in (a)–(d). The students in (a),(b) are distinct. In (c),(d), changing the order of symbols can produce a different string.

(a) Choose three representatives from six students. The representatives have no different roles.

(b) Assign a president, a vice president, and a secretary from six students. Each student may hold at most one position.

(c) Form a length-three string from $\{A, B, C, D, E\}$, allowing repeated symbols.

(d) Form a length-three string from the same symbols, without repeated symbols.

(e) Fix one team of three students from (a). How many assignments to the three positions in (b) use exactly those students? Use this number to relate the answers in (a) and (b).

(f) Give one string counted in (c) but not in (d).

2. Repeated symbols, labeled groups, and binomial coefficients

(a) Count the distinct strings made by using all six letters of `AAABBC` once each.

(b) In your formula for (a), what repeated counting does each factorial in the denominator remove?

(c) Count the allocations of eight distinct students to labeled groups A, B, C of sizes $3, 3, 2$. Order within a group does not matter.

(d) In (c), exchanging the students in groups A and B gives a different allocation. Does your count need a further division by 3!? Answer yes or no, using the group labels as your reason.

(e) Find the coefficient of x^3y^2 in $(2x + y)^5$. Show the choice of factors that produces this term.

(f) Prove $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ by counting all subsets of an n-element set in two ways, including the empty subset.

3. Distributing tokens

Ten tokens are distributed among three labeled boxes A, B, C.

(a) Suppose the tokens are identical and empty boxes are allowed. Let x_1, x_2, x_3 be the box counts. Write the equation and the allowed values of these variables.

(b) Count the distributions in (a) by stars and bars. State the number of stars and the number of bars used in your formula.

(c) Now require at least one token in every box. Use $y_i = x_i - 1$ to count the distributions.

(d) Instead, let all ten tokens be distinct and allow empty boxes. Count the allocations by giving each token a choice of box.

4. Inclusion–exclusion

(a) Prove $|A \cup B| = |A| + |B| - |A \cap B|$ for finite sets. Track the contribution of an element in $A \setminus B$, in $B \setminus A$, and in $A \cap B$.

(b) Let U be all length-four strings over $\{A, B, C\}$, allowing repetition. Calculate $|U|$.

(c) Let M_A, M_B, M_C be the strings missing A, B, C, respectively. Calculate these seven set sizes:

$$|M_A|,\ |M_B|,\ |M_C|,\ |M_A \cap M_B|,\ |M_A \cap M_C|,\ |M_B \cap M_C|,\ |M_A \cap M_B \cap M_C|.$$

(d) Use inclusion–exclusion to calculate $|M_A \cup M_B \cup M_C|$.

(e) Use the complement in U to count the strings containing all three symbols.

5. The pigeonhole principle

(a) State the pigeonhole principle and its generalized version. Specify the number of objects and boxes in each statement.

(b) Prove that at least three of any 25 people have the same birth month. Treat people as objects and months as boxes.

(c) Give a possible allocation of 24 people among the twelve months in which no month contains three people.

(d) Does the conclusion in (b) still hold if birth months are not equally likely? State which step of your proof supports your answer.

6. Sample spaces, events, and probability

Work with a finite or countably infinite sample space Ω.

(a) Define *sample space*, *outcome*, and *event*. For each, specify whether it is a set, an element, or a subset.

(b) Define a probability measure P by stating what objects it assigns numbers to and its three axioms. Include countable additivity for pairwise disjoint events.

(c) Let $p(\omega) = P(\{\omega\})$. Write $P(A)$ as a sum of the probabilities $p(\omega)$ for an event A.

(d) For one roll of a fair die, write the sample space Ω and the event A that the result is even. Calculate $P(A)$.

(e) State the assumptions needed to use $P(A) = |A|/|\Omega|$.

(f) True or false: if a sample space has three outcomes, each must have probability $1/3$.

(g) Prove $P(A^c) = 1 - P(A)$ from the axioms, using $A^c = \Omega \setminus A$. Then apply the result with $A = \Omega$ to find $P(\varnothing)$.

7. Random variables, PMFs, and CDFs

Flip a fair coin twice independently. Let X count heads. Write H for heads and T for tails.

(a) Define a random variable by giving its domain, codomain, and the rule it must provide for each outcome.

(b) For this experiment, list the outcomes and the value of X at each outcome in a two-column table.

(c) Classify each of ω, X, $X(\omega)$, and $\{X = 1\}$ as an outcome, a function, a number, or an event. Write $\{X = 1\}$ as a set of outcomes.

(d) Define the PMF $p_X(x)$, including its nonnegativity and normalization conditions. Write the PMF of this X in a table.

(e) Define the CDF $F_X(t)$. Write this F_X as a piecewise function for all real t, and sketch it with endpoint values marked.

(f) Calculate $P(0 < X \leq 1)$ and $P(0 \leq X \leq 1)$. Identify the outcome that accounts for the difference.

(g) Mark each statement true or false:

(i) A CDF is defined only at values its random variable can take.

(ii) A probability density function (PDF) must be at most 1 everywhere.

(iii) If a random variable has a PDF, its probability at every single point is 0.

8. Conditional probability, independence, and Bayes' theorem

(a) Write the definition of $P(A \mid B)$ and the condition on the denominator for it to be defined.

(b) Write the equation that defines independence of events A and B.

(c) True or false: disjoint events are always independent.

(d) A bag has four red balls and two blue balls. Draw twice, choosing uniformly at each draw. Let R_i mean that draw i is red. Complete the following table for the two sampling models.

Model	$P(R_2 \mid R_1)$	$P(R_2)$	Independent? (yes/no)
With replacement			
Without replacement			

For (e)–(h), a product population has defect rate 2%. A test is positive for 90% of defective products and 5% of nondefective products. Select one product uniformly. Let D be the event that the product is defective and F the event that the test is positive. Thus F^c means a negative test result.

(e) Write the three given rates as probabilities using D and F.

(f) Calculate $P(D \cap F)$, $P(D \cap F^c)$, $P(D^c \cap F)$, and $P(D^c \cap F^c)$.

(g) Use total probability to calculate $P(F)$.

(h) Use Bayes' theorem to calculate $P(D \mid F)$ to three decimal places. State which probability in the given data is $P(F \mid D)$; do not interchange the two conditional probabilities.

9. Bernoulli variables and indicators

Let $P(A) = p$, with $0 \le p \le 1$.

(a) Define the indicator $I_A(\omega)$ with a piecewise formula according to whether $\omega \in A$.

(b) Write $P(I_A = 0)$ and $P(I_A = 1)$. Name the distribution of I_A and its parameter.

(c) Use the definition of expectation to prove $E[I_A] = P(A) = p$.

(d) Use $I_A^2 = I_A$ and the identity in Problem 16(a) to derive $\mathrm{Var}(I_A) = p(1-p)$.

(e) State the value that I_A has with probability one when $p = 0$, and when $p = 1$.

10. Binomial distribution

Perform n independent trials with common success probability $0 < p < 1$. Let I_i indicate success on trial i, and let X count the total successes.

(a) Express X as a sum of the I_i.

(b) State the distribution of X, its parameters, and its possible values.

(c) Derive the PMF of X. Your derivation must identify the probability of one specified sequence with k successes and the number of such sequences.

(d) For $n = 6$ and $p = 1/4$, calculate $P(X = 2)$ and $P(X \geq 1)$.

(e) True or false: the success count must be binomial whenever all trials have success probability p, even if the trials are dependent.

11. Geometric distribution

Transmission attempts are independent, each succeeding with probability $0 < p < 1$. Let T count attempts up to and including the first success.

(a) State the possible values of T.

(b) Derive $P(T = k)$ by specifying the results required on attempts $1, \ldots, k$. Verify normalization with the geometric-series formula.

(c) Describe the event $\{T > m\}$ in terms of the first m attempts, and obtain its probability for every integer $m \geq 0$.

(d) For $p = 1/4$, calculate $P(T = 3)$ and $P(T \leq 3)$.

(e) If a binomial variable counts successes in a fixed number of trials, what does T count instead? State whether the successful attempt is included.

12. Poisson distribution

Assume a constant-rate Poisson arrival model with an average of two requests per minute. The count in t minutes therefore has Poisson parameter $2t$. Let N_1, N_3 count requests in one minute and three minutes.

(a) Write the distribution and parameter of each of N_1, N_3.

(b) Write the Poisson PMF with parameter λ, including the allowed values of the count.

(c) Calculate $P(N_1 = 0)$, $P(N_1 = 3)$, and $P(N_3 \geq 1)$.

(d) In one sentence, describe what λ measures and specify its time interval. Is it a probability that must be at most 1?

(e) True or false: knowing only an average of two requests per minute is enough to determine that the count has a Poisson distribution.

13. Expectation: definition and interpretation

The PMF of X is given below; all other values have probability zero.

x	-1	0	2
$p_X(x)$	$1/4$	$1/4$	$1/2$

(a) Write the definition of $E[X]$ for a discrete variable. Include the absolute-convergence condition for a finite expectation when there are infinitely many possible values.

(b) Calculate $E[X]$ from the displayed PMF.

(c) Calculate $E[X^2]$ and $(E[X])^2$ separately.

(d) A student averages $-1, 0, 2$ with equal weights to find $E[X]$. Identify the weights that should replace those equal weights.

(e) True or false: $E[X]$ must be a value that X can take. Use your answer in (b) to support your judgment.

(f) Does $E[X]$ specify the exact next outcome or a probability-weighted average? Select one interpretation.

14. Proving linearity of expectation

Let X, Y be variables on the same finite sample space Ω, with probabilities $p(\omega)$. Let a, b, c be real constants.

(a) Starting from the PMF definition, group outcomes with the same X-value to prove

$$E[X] = \sum_{\omega \in \Omega} X(\omega) p(\omega).$$

(b) Use (a) to prove $E[aX + bY] = aE[X] + bE[Y]$ by expanding the finite sum. Do not cite linearity as an already proved result.

(c) Did your proof use independence of X, Y? State the step requiring independence, or state that no such step is needed. Check whether the proof still applies to $Y = X$.

(d) Derive $E[aX + c] = aE[X] + c$ from (b) and the expectation of a constant.

(e) Let $I_1, \ldots, I_n$ indicate independent trials with common success probability p. For $Z = \sum_{i=1}^{n} I_i$, use Problem 9(c) and linearity to derive $E[Z] = np$.

15. Variance: definition and meaning

Let $U = 1$ always. Let V be 0 or 2, each with probability 1/2.

(a) Write the definitions of $\mathrm{Var}(X)$ and the standard deviation of X.

(b) Use the definition of variance to prove $\mathrm{Var}(X) \geq 0$.

(c) Calculate $E[U]$, $E[V]$, $\mathrm{Var}(U)$, and $\mathrm{Var}(V)$ directly from their definitions.

(d) Do equal means imply equal variances? Use (c) to answer.

(e) Use linearity to calculate $E[X - E[X]]$ for any variable with finite expectation. Does this quantity distinguish a constant variable from a variable with positive variance?

(f) If X is measured in minutes, state the units of its variance and of its standard deviation.

16. Variance properties

All variables in this problem have finite second moments. Let a, b be real constants.

(a) Expand the square in the definition of variance and use linearity to prove

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2.$$

(b) Starting from the definition of variance, prove

$$\mathrm{Var}(aX + b) = a^2 \,\mathrm{Var}(X).$$

(c) Refute the claim that variances always add. Take $Y = X$ with $\mathrm{Var}(X) > 0$ and compare $\mathrm{Var}(X + Y)$ with $\mathrm{Var}(X) + \mathrm{Var}(Y)$.

(d) State the variance-of-a-sum formula taught in class for independent variables. No proof of this formula is required.

(e) Let $Z = \sum_{i=1}^{n} I_i$, where the I_i are independent Bernoulli(p) variables. Use (d) and Problem 9(d) to derive $\mathrm{Var}(Z) = np(1-p)$.

(f) For $\widehat{p} = Z/n$, calculate $E[\widehat{p}]$, $\mathrm{Var}(\widehat{p})$, and its standard deviation. For fixed $0 < p < 1$, give the factors by which the variance and standard deviation change when n increases from 100 to 400.

17. The hat-check problem

There are $n \geq 2$ people with distinct hats. Return one hat to each person, with all $n!$ assignments equally likely. Let H count people receiving their own hats.

(a) Describe one outcome as an assignment of hats, and specify the sample space.

(b) For person i, define the indicator I_i for receiving their own hat.

(c) Calculate $P(I_i = 1)$. Give either the number of favorable assignments or a symmetry argument.

(d) Express H as a sum of the I_i and use linearity to calculate $E[H]$. The PMF of H is not required.

(e) For $n = 2$, calculate $P(I_1 = 1, I_2 = 1)$ and $P(I_1 = 1)P(I_2 = 1)$. Decide whether I_1, I_2 are independent by comparing these values.

(f) Is independence of the indicators required in (d)? Refer to the condition for linearity established in Problem 14.

18. The coupon collector problem

Each purchase provides one coupon, independently and uniformly drawn from $n \geq 2$ types. Repeated types are possible. Starting with no coupons, let T count purchases until every type has appeared.

For $k = 0, \ldots, n-1$, let W_k count the additional purchases needed to go from k collected types to $k+1$, including the purchase of the new type. You may use $E[W] = 1/p$ for a geometric waiting time counting through the first success. If $p = 1$, that waiting time is always 1.

(a) After k types have been collected, calculate the probability that the next coupon is a new type.

(b) For $1 \leq k \leq n-1$, state the distribution and parameter of W_k and calculate $E[W_k]$. Also state W_0 and $E[W_0]$.

(c) Express T as a sum of the W_k. Use this sum and linearity to derive $E[T]$ as a finite sum in n.

(d) Do you need to prove that the W_k are independent to use linearity in (c)? Refer to Problem 14.

(e) Calculate $E[T]$ for $n = 4$.

(f) Which k gives the largest $E[W_k]$? Relate your answer to the probability of obtaining a new type at that stage.

19. Birthdays in our class

There are 43 students in the Discrete Mathematics for Data Science class. Assume their birthdays are independent and uniformly distributed over 365 dates. Ignore February 29 and birth years: birthdays match when their month and day match. For each answer, give an exact expression and a percentage rounded to two decimal places.

(a) Calculate the probability that all 43 students have different birthdays.

(b) Use the complement of the event in (a) to calculate the probability that at least two students share a birthday.